Propagation of Singularities of Nonlinear Heat Flow in Fissured Media

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Dedicated to the memory of Igor Vladimirovich Skrypnik

1 Introduction

In this paper we investigate the propagation of singularities in a nonlinear parabolic equation with strong absorption when the absorption potential is strongly degenerate following some curve in the (x,t) space. As a very simplified model, we assume that the heat conduction is constant but the absorption of the media depends stronly of the characteristic of the media. More precisely we suppose that the temperature u is governed by the following equation

$$\partial_t u - \Delta u + h(x,t)u^p = 0 \quad \text{in } Q_T := \mathbb{R}^N \times (0,T) \tag{1.1}$$

where $p > 1$ and $h \in C(\overline{Q}_T)$. We suppose that $h(x,t) > 0$ except when (x,t) belongs to some space-time curve Γ given by

$$\Gamma = \{\gamma(\tau) := (x(\tau), t(\tau)) : \tau \in [0,T]\}\,, \tag{1.2}$$

where $\gamma \in C^{0,1}([0,1])$ with $\gamma(0) = 0$ and $t(\tau) > 0$ for $\tau \in (0,T]$. If there holds

$$\int_0^T \int_{B_R} h(x,t)E^q(x,t)dxdt < \infty \tag{1.3}$$

we first show that, for any $k \geq 0$, there exists a unique solution u_k of (1.1) such that $u_k(.,0) = k\delta_0$. Furthermore the mapping $k \mapsto u_k$ is increasing. Because h is continuous and positive outside Γ, we shall show that the set of solutions $\{u_k\}$ remains locally bounded in $Q_T \setminus \Gamma$. Therefore $u_k \uparrow u_\infty$ and u_∞ is a solution of (1.1) in $Q_T \setminus \Gamma$. Then either the singularity of the solution issued from the point $(0,0)$ can propagate along Γ (at least partially), or it remains localized at $(0,0)$. More precisely we show

Theorem A. *Assume γ is C^1 and $t'(\tau) > 0$ for any $\tau \in (0,T]$. The following dichotomy of phenomena occurs*
(i) either $u_\infty(x,t) < \infty$ for all $(x,t) \in Q_T$,
(ii) or there exists $\tau_0 \in (0,T]$ such that

$$\limsup_{(x,t)\to(y,s)} u_\infty(x,t) = \infty \quad \forall (y,s) \in \Gamma, 0 \le s \le \tau_0. \tag{1.4}$$

We first prove that the singularity does not propagate from a point where $t(\tau)$ is decreasing. We prove

Theorem B. *Assume that γ is C^1 and $t'(\tau) < 0$ for all $\tau \in (\tau_0, T]$ for some $\tau_0 > 0$. Then u_∞ remains locally bounded in $Q_T \setminus \gamma((\tau_0,T])$.*

Due to this fact we shall assume first that the t variable is increasing along Γ, in such a case we can assume that τ is a function of t and, up to a change of parameter, that $\tau = t$ and

$$\Gamma = \{(x(t),t) : t \in [0,T]\}, \tag{1.5}$$

where $\gamma \in C^{0,1}(0,T]$ satisfies $\gamma(0) = 0$. In order the singularity to propagate along Γ, h must be very flat near this curve Γ. If we define the parabolic distance between (x,t) and Γ by

$$d_P(x,t);\Gamma) = \inf\{d_P(x,t);(y,s)) : (y,s) \in \Gamma, s \le t\}$$

where $d_P(x,t);(y,s) = |x-y| + \sqrt{t-s}$, for $t \ge s$ and write

$$h(x,t) = e^{-\ell(d_P(x,t);\Gamma))},$$

where ℓ is a positive nonincreasing function. In the next result we shows that propagation of singularity along Γ occurs

Theorem C. *Assume $t \mapsto x(t)$ belongs to $W^{2,\infty}_{loc}[0,\infty)$ and*

$$\liminf_{t\to 0} t^2\ell(t) > 0. \tag{1.6}$$

Then $\lim_{(x,t)\to(y,s))} u_\infty(x,t) = \infty$ for all $(y,s) \in \Gamma$.

The last section is devoted to the case where the curve of degeneracy Γ is a straight line contained in the initial plan. We set $d_\infty((x,t),\Gamma) = \max\{\sqrt{t}, x'\}$ if $\mathrm{dist}\,(x,\Gamma) = x'$ and we write

$$h(x,t) = e^{-\ell(d_\infty(x,t);\Gamma))}$$

Up to a rotation, we can suppose that Γ is the x_1 axis and denote $x = (x_1, x')$ the component in $\mathbb{R} \times \mathbb{R}^{N-1}$. Then we prove the following

Theorem D. *Assume*

$$\liminf_{t\to 0} t^2\ell(t) > 0. \tag{1.7}$$

Then $\lim_{(x,t)\to(y_1,0))} u_\infty(x,t) = \infty$ for all $y_1 \in \mathbb{R}$.

Even if this model is a very simplified version of the heat propagation in a fissured absorbing media, it gives interesting insight of the propagation phenomenon which can occur. It is also a starting point for studying other type of propagation of singularities in nonlinear diffusion equations. In a forthcoming article we shall consider the case where the degeneracy line is a surface in $\mathbb{R}^N \times [0,\infty)$ with only one contact point with $t = 0$ at $(0,0)$.

2 Preliminaries and basic estimates

Through out this section we assume that $h \in C(\mathbb{R}^N \times [0,\infty))$ is nonnegative. Since $E(x,t)$ is a supersolution for (1.1), the following result holds [10, Theorem 6.12]

Proposition 2.1 *Assume $q > 1$ and (1.3) holds. Then for any $k > 0$ there exists a unique $u = u_k \in C(\overline{Q}_T \setminus \{(0,0)\}) \cap L^1(Q_T)$, such that $h|u|^q \in L^1(Q_T)$, satisfying*

$$\iint_{Q_T} (-u\partial_t\zeta - \Delta\zeta + h(x,t)|u|^{q-1}u\zeta)dx\,dt = k\zeta(0,0) \tag{2.1}$$

for all $\zeta \in C^{2,1}(\overline{Q}_T)$ which vanishes at $t = T$.

More generaly, if $\mu \in \mathfrak{M}(\mathbb{R}^N)$ and $\nu \in \mathfrak{M}(\mathbb{R}^N \times (0,T))$ are two positive bounded measures, the solution $v = v_{\mu,\nu}$ of

$$\left\{\begin{array}{ll} \partial_t v - \Delta v = \nu & \text{in } \mathbb{R}^N \times [0,T) \\ v(.,0) = \mu & \text{in } \mathbb{R}^N, \end{array}\right. \tag{2.2}$$

is expressed by

$$v_{\mu,\nu}(x,t) = \frac{1}{(4\pi t)^{N/2}}\int_{\mathbb{R}^N} e^{-4|x-y|^2/4t}d\mu(y) + \int_0^t \frac{1}{(4\pi(t-s))^{N/2}}\int_{\mathbb{R}^N} e^{-4|x-y|^2/4(t-s)}d\nu(y)ds. \tag{2.3}$$

Actually, by direct adaptation to the parabolic case of [10, Theorem 4.2], combined with [10, Theorem 6.12], one can proves that, for any bounded Radon measures μ on $\mathbb{R}^N \times (0,\infty)$ and ν on $\mathbb{R}^N$ which satisfy

$$\iint_{Q_T} v^q_{\mu_s,\nu_s} h(x,t)dxdt < \infty \tag{2.4}$$

where μ_s and ν_s are the singular parts (with respect to respective Lebesgue measures) μ and ν respectively, there exists a unique weak solution u to problem

$$\left\{\begin{array}{ll} \partial_t u - \Delta u + h(t,x)|u|^{q-1}u = \nu & \text{in } \mathbb{R}^N \times [0,T) \\ u(.,0) = \mu & \text{in } \mathbb{R}^N, \end{array}\right. \tag{2.5}$$

The next result is an adaptation of Brezis-Friedman a priori estimate [1].

Proposition 2.2 *Let $Q = Q^{r,a}_{t_0,t_1} := B_r(a) \times (t_0, t_1)$ for some $a \in \mathbb{R}^N$, $t_1 > t_0 \geq 0$ and $r > 0$ and assume $\beta = \min\{h(x,t) : (x,t) \in \overline{Q}\} > 0$. Then any solution of (1.1) in Q satisfies*

$$|u(x,t)| \leq \frac{C}{\beta^{1/(q-1)}} \left(\frac{1}{t-t_0} + \frac{1}{(r-|x-a|)^2} \right)^{1/(q-1)} \qquad \forall (x,t) \in Q, \tag{2.6}$$

for some $C = C(N,q) > 0$.

Proof. The maximal solution of the parabolic equation $y' + \beta y^q = 0$ on $0, \infty)$ is expressed by

$$y_{_M}(t) = \left(\frac{1}{\beta(q-1)t} \right)^{1/(q-1)}. \tag{2.7}$$

By Keller-Osserman estimate, the maximal solution $v_{_M}$ of $-\Delta v + \beta |u|^{q-1} u = 0$ in B_r, satisfies

$$v_{_M}(x) \leq C \left(\frac{1}{\beta(r-|x|)^2} \right)^{1/(q-1)} \tag{2.8}$$

in B_r. Since $y_{_M}(t-t_0) + v_{_M}(x-a)$ is a supersolution of (1.1) in Q which blows up on the parabolic boundary, an easy approxition argument (just replacing r by $\{r_n\} \uparrow r$ and t_0 by $\{t_n\} \downarrow t_0$) leads us to (2.6). $\square$

Proposition 2.3 *Let $0 < r_0 < r_1$, $0 < t_0 < t_1$ and $\Theta := \Theta^{r_0,r_1,a}_{t_0,t_1} := Q^{r_1,a}_{t_0,t_1} \setminus Q^{r_0,a}_{t_0,t_1}$ for some $a \in \mathbb{R}^N$ and assume $\beta = \min\{h(x,t) : (x,t) \in \overline{\Theta}\} > 0$. Then any solution of (1.1) in $Q^{r_1,a}_{t_0,t_1}$ such that $|u(x,t_0)| \leq \mu$ for $x \in B_{r_1}(a)$ satisfies*

$$|u(x,t_1)| \leq \mu + \frac{C}{(\beta(r_1-r_0)^2)^{1/(q-1)}} \qquad \forall x \in B_{r_0}(a), \tag{2.9}$$

for some $C = C(N,q) > 0$.

Proof. Let $b \in \mathbb{R}^N$ such that $|b-a| = \frac{r_0+r_1}{2}$ and v_M the maximal solution of $-\Delta v + \beta |u|^{q-1} u = 0$ in $B_{\frac{r_1-r_0}{2}}(b)$. Then

$$v_M(x) \leq \frac{C}{\left(\beta \left(\frac{r_1-r_0}{2} - |x-b|\right)\right)^{2/(q-1)}} \qquad \forall x \in B_{\frac{r_1-r_0}{2}}(b).$$

Since $v_M + \mu$ is a super solution of (1.1) in $Q^{\frac{r_1-r_0}{2},b}_{t_0,t_1}$ which dominates u at $t = t_0$ and for $|x-b| = \frac{r_1-r_0}{2}$, it follows that $u(x,t) \leq v_M(x,t) + \mu$ in $Q^{\frac{r_1-r_0}{2},b}_{t_0,t_1}$. In particular, if $x = b$, we get

$$u(b,t) \leq \mu + \frac{C}{\left(\beta \left(\frac{r_1-r_0}{2}\right)\right)^{2/(q-1)}}$$

This estimate is valid for any $b \in \mathbb{R}^N$ with $|b-a| = \frac{r_0+r_1}{2}$. Since u is a subsolution of the heat equation in $Q^{r_0,a}_{t_0,t_1}$, (2.9) follows by the maximum principle. $\square$

The previous estimates are based upon constructions of supersolutions in cylinders. In the next result we construct estimates in tubular neighborhood of Γ. If $t(\tau)$ is increasing, we can take $\tau = t$ and

$$\Gamma = \gamma([0,T]) := \{(x(t),t) : t \in [0,T]\} \tag{2.10}$$

Proposition 2.4 *Assume Γ is C^1 and parametrized by t as in (2.10), and for $\epsilon > 0$ denote*

$$T_s^{\Gamma_0^T} := \{(x,t) : 0 \le t \le T,\, |x - x(t)| < s\}.$$

For $1 \ge r_1 > r_0 > 0$ we set $\eta = \min\left\{h(x,t) : (x,t) \in T_{r_1}^{\Gamma_0^T} \setminus T_{r_0}^{\Gamma_0^T}\right\}$. Then there exists a constant C depending on $c := \max\{|x'(t)| \; : 0 \le t \le T\}$ such that if

$$u(x,0) \le m + \frac{C_2}{(r_1 - r_0)^{\frac{2}{q-1}}} \qquad \forall x \in \overline{B_{\frac{r_1+r_0}{2}}(x(0))}, \tag{2.11}$$

then

$$u(x,t) \le \left(\frac{m^{q-1}}{1 + \eta(q-1)m^{q-1}t}\right)^{\frac{1}{q-1}} + \frac{C_2}{(r_1 - r_0)^{\frac{2}{q-1}}} \qquad \forall t \in [0,T] \text{ and } x \in \overline{B_{\frac{r_1+r_0}{2}}(x(t))} \tag{2.12}$$

Proof. Consider the change of space variable $x = y + x(t)$ and $u(x,t) = v(y,t)$. Then v satisfies

$$\partial_t v - \Delta v + \langle x'(t), \nabla v\rangle + h(y + x(t), t)v^q = 0 \qquad \text{in } Q_{0,t(T)}^{r_0,r_1}. \tag{2.13}$$

Thus

$$\partial_t v - \Delta v - c|\nabla v| + \eta v^q \le 0 \qquad \text{in } Q_{0,t(T)}^{r_0,r_1}.$$

In the ball $B_{\frac{r_1-r_0}{2}}(z)$ where $|z| = \frac{r_1+r_0}{2}$ It is standard easy to construct a radial function ψ satisfying

$$-\Delta\psi - c|\nabla\psi| + \eta\psi^q \ge 0 \qquad \text{in } B_{\frac{r_1-r_0}{2}}(z) \tag{2.14}$$

under the form

$$\psi(y) = C\frac{\rho^{\frac{2}{q-1}}}{(\rho^2 - |y-z|^2)^{\frac{2}{q-1}}}$$

where $\rho = \frac{r_1-r_0}{2}$ and $C = C(N,q,c,\eta)$. Therefore

$$\psi(z) = \frac{C_1}{\rho^{\frac{2}{q-1}}} = \frac{C_2}{(r_1 - r_0)^{\frac{2}{q-1}}}$$

The solution $\phi_m = \phi$ of

$$\begin{cases} \phi' + \eta\phi^q = 0 & \text{on } (0,T) \\ \phi(0) = m > 0 \end{cases} \tag{2.15}$$

is expressed by

$$\phi_m(t) = \left(\frac{m^{q-1}}{1+\eta(q-1)m^{q-1}t}\right)^{\frac{1}{q-1}}.$$

Consequently, the function $\phi_m(t)+\psi(y)$ is a supersolution for (2.13) in $Q_{0,t(T)}^{r_0,r_1}$ which domintates v at $t=0$ and on the lateral boundary. Therefore it is larger than v and in particular

$$v(z,t) \leq \phi_m(t) + \psi(z). \tag{2.16}$$

Consequently

$$u(x,t) \leq \phi_m(t) + \psi(z) \qquad \forall t \in (0,T) \text{ and } |x - x(t)| = \frac{r_1 + r_0}{2}. \tag{2.17}$$

We derive (2.12) by the maximum principle. □

3 Geometric obstruction to propagation

We assume that h vanishes on a continuous curve $\Gamma \subset \mathbb{R}^N \times [0,T)$ defined by parametrisation

$$\Gamma = \{\gamma(t) := (x(\tau), t(\tau) : \tau \in [0,T]\} \tag{3.1}$$

issued from $(0,0)$ (i.e. $x(0), t(0) = (0,0)$ with $t(\tau) > 0$ if $\tau \in (0,T)$. and $\tau \mapsto \gamma(\tau)$ is Lipschitz with no self intersection, which means that $\tau \mapsto \gamma(\tau)$ is one to one.

Proof of Theorem A. Since t' is continuous and increasing, we apply Proposition 2.4 with 0 replaced by τ_0, we set $t_0 = t(\tau_0)$ and write Γ under the form (2.10). If we assume that

$$\limsup_{(x,t)\to(x(t_0),t_0)} u(x,t) < \infty,$$

there exists $r_1 > 0$ and $\mu > 0$ such that $u(x,t_0) \leq \mu$ if $x \in B_{r_1}(x(t_0))$. Then, for $0 < r_0 < r_1$ there exists $m > 0$ such that (2.10) is verified. Thus (2.13) holds. This implies that the blow-up set of u along Γ is empty if $t \geq t_0$. □

Proof of Theorem B. The proof is based upon the same ideas than in Theorem A above except that we only study the part of Γ between τ_0 and T, where $t'(\tau) < 0$. Let $t_0 = t(\tau_0)$ and $t^* = t(T)$. We parametrized Γ by t between t^* and t_0, thus

$$\gamma([\tau_0, T]) = \{(t, x(t)) : t^* \leq t \leq t_0\}.$$

For $s > 0$ and $t^* < t' \leq t_0$, we set $T_s^{\Gamma_{t*}^{t'}} := \{(x,t) : t^* \leq t \leq t_0,\ |x - x(t)| < s\}$, then, for $t^* < t_1 < t_0$, there exists $r_1 > 0$ and $\tau_1 \in (\tau_0, T)$ such that $t_1 = t(\tau_1)$ and $\Gamma \cap T_{r_1}^{\Gamma_{t*}^{t_1}} = \gamma([\tau_1, T])$. If $t^* = 0$, then $u(x,t^*) = 0$ for $|x - x(t^*)| \leq r_1$. If $t^* > 0$, then $h(x,t) > 0$ in the cylinder $Q_{0,t^*}^{r_1,x(t^*)}$, thus there exists $\beta > 0$ such that $\inf\left\{h(x,t) : (x,t) \in Q_{0,\frac{t^*}{2}}^{r_1,x(t^*)}\right\} = \beta$. Up to replacing r_1 by some $r_1' > r_1$ such that $\Gamma \cap \overline{Q_{0,\frac{t^*}{2}}^{r_1,x(t^*)}} = \emptyset$, we obtain that $u_\infty(x,t^*) \leq \mu$ for

all $x \in \overline{B_{r_1}(x(t^*))}$ from Proposition 2.2. In both cases $u_\infty(x,t^*)$ is bounded in $\overline{B_{r_1}(x(t^*))}$. Replacing $(0,T)$ by $(t^*,t_1))$ in Proposition 2.4, it follows that u_∞ remains bounded in $T^{\Gamma^{t_1}_{t^*}}_{\frac{r_1+r_0}{2}}$ for some $0 < r_0 < r_1$. Since $t_1 < t_0$ is arbitrary and u_∞ is locally bounded in $Q_T \setminus \Gamma$, the proof follows. $\square$

In the next case the monotonicity of $\tau \mapsto t(\tau)$ is replaced by a *box-assumption*.

Proposition 3.1 *Assume that γ is continuous and there exists $a \in \mathbb{R}^N$, $r_0 > 0$ and $\tau_0 \in (0,T)$ such that $t(T) \leq t(\tau_0)$ and $\gamma([\tau_0,T]) \subset B_{r_0}(a) \times [t(T), t(\tau_0)]$. Then u_∞ is bounded in $\overline{B}_{r_0}(a) \times [t(T), t(\tau_0)]$.*

Proof. There exist $r_0' < r_0$ and $\beta > 0$ such that $\gamma([\tau_0,T]) \subset B_{r_0'}(a) \times [t(T), t(\tau_0)]$ and $\min\left\{h(x,t) : (x,t) \in \Theta^{r_0',r_0}_{t(T),t(\tau_0)}\right\} = \beta$. Therefore the conclusion follows from Proposition 2.3. $\square$

In the next case we show that non-propagation of singularities may occur even if $\tau \mapsto t(\tau)$ is increasing after some τ_0, provided there is a local maximum in $(0,\tau_0)$. We put $\Gamma_{\tau_0} = \gamma([0,\tau_0])$

Theorem 3.2 *Let γ be C^1 and $\gamma'(\tau) \neq 0$ on $[0,T]$. Assume there exist $\tau_0 > 0$, $a \in \mathbb{R}^N$ and $r > 0$ such that $t(\tau) \leq t(\tau_0)$ on $[0,\tau_0]$, $\tau \mapsto t(\tau)$ is decreasing on $(\tau_0,\tau_0+\delta)$ for some $\delta > 0$, $\gamma((\tau_0,T]) \subset Q^{0,\infty}_{r,a}$, $\Gamma_{\tau_0} \cap \overline{Q^{0,\infty}_{r,a}} = \{\gamma(\tau_0)\}$ and for any $\tau \in (\tau_0,T]$, $|x(\tau) - a| < r$. Then u_∞ is locally bounded in $Q \setminus \Gamma_{\tau_0}$, where $\Gamma_{\tau_0} := \gamma([0,\tau_0])$.*

Proof. Since $t(\tau)$ is decreasing on $(\tau_0,\tau_0+\delta)$, for any $\tau_1 \in (\tau_0,\tau_0+\delta)$, the set of $\tau \in (\tau_1,T]$ such that $t(\sigma) < t(\tau_1)$ for all $\sigma \in (\tau_1,\tau)$ is not empty. Its upper bound τ_1^* is less or equal to T and $t(\tau_1^*) = \min\{t(\tau_1), t(T)\}$. If $\lim_{\tau_1 \to \tau_0} t(\tau_1^*) = t(T)$, then $t(\tau_0) = t(T)$, the *box-assumption* holds and the conclusion follows from Proposition 3.1. If $t(T) > t(\tau_0)$, then $t(\tau_1) = t(\tau_1^*) < t(T)$ for any $\tau_1 \in (\tau_0,\tau_0+\delta)$. Since $\lim_{\tau_1 \to \tau_0} t(\tau_1) = t(\tau_0)$ and γ is continuous with $\gamma((\tau_0,T]) \subset Q^{0,\infty}_{r,a}$, there some fixed constants $\lambda > 0$ and $\rho > 0$ such that $x(\tau) \in B_\rho(x(\tau_1))$, for any $\tau \in [\tau_1,T]$ verifying $t(\tau) \leq t(\tau_1) + \lambda$. This means that the part of Γ starting from $\gamma(\tau_1)$ for which $t(\tau)$ belongs to $(t(\tau_1), t(\tau_1)+\lambda)$ remains in $B_\rho(x(\tau_1))$. Moreover we can assume that $t(\tau_1)+\lambda > t(\tau_0)$. By restricting ρ, we can assume that $B_\rho(x(\tau_1)) \subset B_{r'}(a)$. Since $u_k(x,t(\tau_1)$ is uniformly bounded when $x \in B_\rho(x(\tau_1))$, it follows from Proposition 2.3 that for any $\rho' \in (0,\rho)$, u_k remains uniformly bounded in $Q^{t(\tau_1),t(\tau_1)+\lambda}_{\rho',x(\tau_1)}$. Moreover, for any compact subset $K \Subset B^c_{\rho'}(x_{\tau_1})$, there holds

$$u_k(x, t(\tau_1)+\lambda) \leq \frac{C}{(\lambda\beta)^{1/(q-1)}},$$

where $\beta = \min\{h(x,t) : (x,t) \in K \times [t(\tau_1), t(\tau_1)+\lambda]\}$. Iterating this construction, we can construct a finite number of cylinders $Q^{t(\tau_1)+j\lambda, t(\tau_1)+(j+1)\lambda}_{\rho',a_j}$ containing Γ and in which u_k remains uniformly bounded. Since local uniform boundedness holds also outside such cylinders, the proof follows. $\square$

Remark. In full generality we conjecture that if γ is C^1 with $\gamma'(\tau) \neq 0$ and there exists $\tau_0 \in (0,T)$ such that $t(\tau)$ admits a local strict maximum on the right at $\tau_0 \in (0,T)$, then u_∞ is locally bounded $Q_T \setminus \Gamma_{\tau_0}$.

4 Propagation of singularities in the space

In this section, we assume that the degeneracy curve Γ is parametrized by the variable $t \in [0,T]$ and defined by (2.10) with $|x(t)| > 0$ if $t > 0$. We denote by

$$d_P[(x,t),(y,s)] := |x-y| + \sqrt{t-s} \quad \text{if } t \geq s$$

the parabolic distance and we assume that $h(x,t)$ depends on $d_P[(x,t),\Gamma]$ under the following form

$$h(x,t) = e^{-\ell(d_P[(x,t),\Gamma])} \tag{4.1}$$

where $\ell \in C([0,\infty))$ is positive, nonincreasing and $\lim_{r\to 0} \ell(r) = \infty$. For $\epsilon > 0$, we recall $T_\epsilon^{\Gamma_0^T}$ denotes the ϵ -spherical tubular neighborhood of Γ between $t = 0$ and $t = T$ defined by

$$T_\epsilon^{\Gamma_0^T} := \{(x,t) \in Q_T : |x - x(t)| < \epsilon\}. \tag{4.2}$$

The basis of $T_\epsilon^{\Gamma_0^T}$, in $\mathbb{R}^N \times \{0\}$, is the ball B_ϵ. Since $d_P[(x,t),\Gamma] \leq |x-x(t)|$, $d_P[(x,t),\Gamma] \leq \epsilon$ in $T_\epsilon^{\Gamma_0^T}$ and $\ell(\epsilon) \leq \ell(d_P[(x,t),\Gamma])$. Then, u_∞ is bounded from below in Γ_ϵ by the solution v_ϵ of

$$\left\{\begin{aligned} \partial_t v_\epsilon - \Delta v_\epsilon + e^{-\ell(\epsilon)} v_\epsilon^p &= 0 && \text{in } T_\epsilon^{\Gamma_0^T} \\ v_\epsilon &= 0 && \text{in } \partial_{\mathcal{P}} T_\epsilon^{\Gamma_0^T} \\ v_\epsilon &= \infty\delta_0 && \text{in } B_\epsilon, \end{aligned}\right. \tag{4.3}$$

where

$$\partial_{\mathcal{P}} T_\epsilon^{\Gamma_0^T} := \{(x,t) : |x - x(t)| = \epsilon\}$$

is the lateral parabolic boundary of $T_\epsilon^{\Gamma_0^T}$. The formal aspect comes from the fact that the existence of v_ϵ has to be proved.

For the sake of simplicity we assume that $1 < p < 1 + \frac{2}{N}$, the case $p \geq 1 + \frac{2}{N}$ needing a simple adaptation of the type which is developed in the proof of Theorem D-Case 2.

We consider the following change of variable $x = y + x(t)$ and $v_\epsilon(x,t) = \tilde{v}_\epsilon(y,t)$. Then $\tilde{v}_\epsilon$ satisfies in $Q_\tau^{B_\epsilon} := B_\epsilon \times (0,\tau)$

$$\left\{\begin{aligned} \partial_t \tilde{v}_\epsilon - \Delta \tilde{v}_\epsilon + \langle x'(t)|\nabla \tilde{v}_\epsilon\rangle + e^{-\ell(\epsilon)} \tilde{v}_\epsilon^p &= 0 && \text{in } Q_\tau^{B_\epsilon} \\ \tilde{v}_\epsilon &= 0 && \text{in } \partial_{\mathcal{P}} Q_\tau^{B_\epsilon} \\ \tilde{v}_\epsilon &= \infty\delta_0 && \text{in } B_\epsilon. \end{aligned}\right. \tag{4.4}$$

In particular $u(x(\alpha),\alpha) \geq v_\epsilon(x(\alpha),\alpha) = \tilde{v}_\epsilon(0,\alpha)$. We set

$$\omega_\epsilon(x,t) = \epsilon^{2/(p-1)} e^{-\ell(\epsilon)/(p-1)} \tilde{v}_\epsilon(\epsilon x, \epsilon^2 t),$$

and $x_\epsilon(t) = \epsilon^{-1}x(\epsilon^2 t)$. Then ω_ϵ satisfies

$$\left\{\begin{array}{rl} \partial_t\omega_\epsilon - \Delta\omega_\epsilon + \langle x'_\epsilon(t)|\nabla\omega_\epsilon\rangle + \omega_\epsilon^p = 0 & \text{in } Q^{B_1}_{\epsilon^{-2}\tau} \\ \omega_\epsilon = 0 & \text{in } \partial_{\mathcal{P}} Q^{B_1}_{\epsilon^{-2}\tau} \\ \omega_\epsilon = \infty\delta_0 & \text{in } B_1, \end{array}\right. \tag{4.5}$$

and for any $\alpha > 0$

$$u_\infty(x(\alpha), \alpha) \geq \epsilon^{-2/(p-1)} e^{\ell(\epsilon)/(p-1)} \omega_\epsilon(0, \epsilon^{-2}\alpha). \tag{4.6}$$

Therefore, the problem is reduced to showing that

$$\lim_{\epsilon\to 0} \epsilon^{-2/(p-1)} e^{\ell(\epsilon)/(p-1)} \omega_\epsilon(0, \epsilon^{-2}\alpha) = \infty. \tag{4.7}$$

We associate the following parabolic equation

$$\left\{\begin{array}{rl} \partial_t v - \Delta v + \langle \beta(t)|\nabla v\rangle + v^p = 0 & \text{in } Q^{B_1}_\infty \\ v = 0 & \text{in } \partial_{\mathcal{P}} Q^{B_1}_\infty \\ v(y, 0) = v_0 & \text{in } B_1. \end{array}\right. \tag{4.8}$$

Existence of solution is classical when $\beta \in L^\infty_{loc}([0, \infty))$ (see [4], [5]).

Proposition 4.1 *Let $\beta \in L^\infty_{loc}([0, \infty))$. Then the following estimates holds*

$$\|v(., t)\|_{L^2} \leq e^{-\lambda_0 t} \|v(., 0)\|_{L^2} \tag{4.9}$$

where λ_0 is the first eigenvalue of $-\Delta$ in B_1, and, for some $C = C(N) > 0$,

$$\|v(., t)\|_{L^\infty} \leq C \min\{t^{-N/4}, e^{-\lambda_0 t}\} \|v(., 0)\|_{L^2}. \tag{4.10}$$

Proof. Since for any continuous function $r \mapsto g(r) = G'(r)$, we have

$$\int_{B_1} \langle \beta(t)|\nabla v\rangle g(v)dx = \int_{B_1} \langle \beta(t)|\nabla G(v)\rangle dx = -\int_{B_1} G(v)\, div\beta(t) dx = 0,$$

we have

$$2^{-1}\frac{d}{dt}\int_{B_1} v^2 dx + \int_{B_1} |\nabla v|^2 dx \leq 0 \Longrightarrow \frac{d}{dt}\int_{B_1} v^2 dx \leq -2\lambda_0 \int_{B_1} v^2 dx.$$

Thus (4.9) follows. Furthermore, for any $q > 1$,

$$\frac{1}{q+1}\frac{d}{dt}\int_{B_1} |v|^{q+1} dx + \frac{4q}{(q+1)^2}\int_{B_1} |\nabla v^{(q+1)/2}| dx \leq 0,$$

it follows that $t \mapsto \|v(., t)\|_{L^{q+1}}$ is decaying. Therefore, using Gagliardo-Nirenberg estimate,

$$\frac{d}{dt}\|v\|^{q+1}_{L^{q+1}} + \frac{4qC}{q+1}\|v\|^{q+1}_{L^{(q+1)N/(N-2)}} \leq 0.$$

We assume $N \geq 3$, the cases $N = 1, 2$ needing a simple modification. Thus, for any $t > s > 0$,

$$\frac{4qC(t-s)}{q+1} \|v(.,t)\|^{q+1}_{L^{(q+1)N/(N-2)}} \leq \|v(.,s)\|^{q+1}_{L^{q+1}}.$$

By a standard use of Moser iterative scheeme, we derive

$$\|v(.,t)\|_{L^\infty} \leq Ct^{-N/4} \|v(.,0)\|_{L^2}. \tag{4.11}$$

Consequently, for any $0 < s < t$, we have

$$\begin{aligned}\|v(.,t)\|_{L^\infty} &\leq C(t-s)^{-N/4} \|v(.,s)\|_{L^2} \\ &\leq C(t-s)^{-N/4} e^{-\lambda_0 s} \|v(.,0)\|_{L^2}.\end{aligned}$$

Noting that, if $t > N/4\lambda_0$,

$$\min\{(t-s)^{-N/4} e^{-\lambda_0 s} 0 < s < t\} = e^{N/4}(4\lambda_0/N)^{N/4} e^{-\lambda_0 t},$$

we derive (4.10). □

Proposition 4.2 *Let Ω be a bounded open domain in $\mathbb{R}^N$, $\beta \in \mathbb{R}^N$ and L_β the operator $v \mapsto -\Delta v + \langle \beta | \nabla v \rangle$. Then the spectrum of L_β in $H^{1,0}(\Omega)$ is the given by*

$$\sigma(L_\beta) = \left\{ \lambda + \frac{|\beta|^2}{4} : \lambda \in \sigma(L_0) \right\}. \tag{4.12}$$

Proof. Put $v(x) = e^{\frac{1}{2}\langle \beta | x \rangle} w(x)$. Then

$$\nabla v(x) = e^{\frac{1}{2}\langle \beta | x \rangle} \left(\nabla w + \frac{w}{2} \beta \right)$$

$$\Delta v(x) = e^{\frac{1}{2}\langle \beta | x \rangle} \left(\Delta w + \langle \beta | \nabla w \rangle + \frac{|\beta|^2}{4} w \right)$$

Thus

$$L_\beta v = e^{\frac{1}{2}\langle \beta | x \rangle} \left(-\Delta w + \frac{|\beta|^2}{4} w \right), \tag{4.13}$$

and the proof follows. □

In the sequel we denote by λ_β the first eigenvalue of L_β, thus $\lambda_\beta = \lambda_0 + \frac{|\beta|^2}{4}$. If ψ_β is a corresponding positive eigenfunction, then

$$\psi_\beta(x) = e^{\frac{1}{2}\langle \beta | x \rangle} \psi_0(x)$$

where ψ_0 is a positive first eigenfunction of $-\Delta$ in $H^{1,0}(\Omega)$

Proposition 4.3 *Under the assumptions of Proposition 4.2 and for $p>1$, we denote by v be the solution of*

$$\left\{\begin{aligned} \partial_t v-\Delta v+\langle\beta|\nabla v\rangle+v^p=0 \qquad & \textit{in } Q_\infty^{B_1}\\ v=0 \qquad & \textit{in } \partial_p Q_\infty^{B_1}\\ v(y,0)=v_0 \qquad & \textit{in } B_1. \end{aligned}\right. \tag{4.14}$$

where $v_0\in L^2(B_1)$ is nonnegative. Then there exists some $c=c(v_0)>0$ such that

$$\lim_{t\to\infty} e^{\lambda_\beta t}v(.,t)=c\psi \tag{4.15}$$

uniformly in B_1.

Proof. We write $v(x,t)=e^{\frac{1}{2}\langle\beta|x\rangle}w(x,t)$, thus (4.14) turns into

$$\partial_t w-\Delta w+\frac{|\beta|^2}{4}w+e^{\frac{p-1}{2}\langle\beta|x\rangle}w^p=0. \tag{4.16}$$

The proof of [3, Th 3.1] applies easily and the result follows. □

Proposition 4.4 *Assume $\beta\in W^{1,\infty}_{loc}([0,\infty))$. For $\tau>1$, we set $\sup_{1\leq t\leq\tau}|\beta(t)|=\beta_\tau$ and $\sup_{1\leq t\leq\tau}|\beta'(t)|=\delta_\tau$. If v is the solution of (4.8) where $v_0\in L^2(B_1)$ is nonnegative, we denote $\sigma(\tau)=\sup\left\{e^{\frac{p-1}{2}|\beta(t)|}w^{p-1}(x,t):(x,t)\in B_1\times[1,\tau]\right\}$. Then if $v(.,1)$ satisfies $c_1\psi_0\leq v(.,1)$, there holds*

$$v(x,t)\geq c_1e^{-(\lambda_0+\frac{\beta_\tau^2}{4}+\frac{\delta_\tau}{2}+\sigma_\tau)(t-1)}\psi_0(x)\quad\forall(x,t)\in B_1\times[1,\tau] \tag{4.17}$$

Proof. Let $w(x,t)=e^{-\frac{1}{2}\langle\beta(t)|x\rangle}v(x,t)$. Then

$$\partial_t w-\Delta w+\left(\frac{|\beta|^2}{4}+\frac{1}{2}\langle\beta'|x\rangle+e^{\frac{p-1}{2}\langle\beta(t)|x\rangle}w^{p-1}\right)w=0. \tag{4.18}$$

Since $|x|\leq 1$ there holds in $B_1\times[1,\tau]$

$$\frac{|\beta(t)|^2}{4}+\frac{1}{2}\langle\beta'(t)|x\rangle+e^{\frac{p-1}{2}\langle\beta(t)|x\rangle}w^{p-1}\leq\frac{\beta_\tau^2}{4}+\frac{\delta_\tau}{2}+\sigma_\tau. \tag{4.19}$$

Therefore

$$\partial_t w-\Delta w+\left(\frac{\beta_\tau^2}{4}+\frac{\delta_\tau}{2}+\sigma_\tau\right)w\geq 0\quad\text{in } B_1\times[1,\tau]. \tag{4.20}$$

Since $(x,t)\mapsto e^{-(\lambda_0+\frac{\beta_\tau^2}{4}+\frac{\delta_\tau}{2}+\sigma_\tau)(t-1)}\psi_0(x)$ satisfies the equation associated to (4.20), (4.17) follows. □

Proof of Theorem C. Step 1: Initialization of the blow-up. With our previous notations, $\beta(t)=\beta_\epsilon(t)=x'_\epsilon(t)=\epsilon x'(\epsilon^2t)$ and $\beta'_\epsilon(t)=\epsilon^3x''(\epsilon^2t)$. Since x'_ϵ is locally bounded, it follows

by Hopf lemma that there exists $c_1 > 0$ such that $\omega_\epsilon(.,1) \geq c_1\psi_0$. We take $\tau = \epsilon^{-2}\alpha$ where $\alpha > 0$ is fixed. Then

$$\beta_\tau = \sup\{|\epsilon x'(\epsilon^2 t)| : 1 \leq t \leq \epsilon^{-2}\alpha\} = \epsilon \sup\{|x'(t)| : \epsilon^2 \leq t \leq \alpha\},$$

$$\delta_\tau = \sup\{|\epsilon^3 x''(\epsilon^2 t)| : 1 \leq t \leq \epsilon^{-2}\alpha\} = \epsilon^3 \sup\{|x''(t)| : \epsilon^2 \leq t \leq \alpha\}.$$

Therefore

$$\epsilon^{-\frac{2}{p-1}} e^{\frac{\ell(\epsilon)}{p-1}} \omega_\epsilon(0, \epsilon^{-2}\alpha) \geq c_1' e^{A(\epsilon)} \psi_0(0) \tag{4.21}$$

where

$$A(\epsilon) := -\frac{2}{p-1}\ln\epsilon + \frac{\ell(\epsilon)}{p-1} - \left(\lambda_0 + \frac{\beta_\tau^2}{4} + \frac{\delta_\tau}{2} + \sigma_\tau\right)\frac{\alpha}{\epsilon^2} \tag{4.22}$$

Since $\liminf_{\epsilon\to 0} \epsilon^2\ell(\epsilon) > 0$ there exists $\alpha_0 > 0$ such that for any $0 < \alpha < \alpha_0$, there holds $\lim_{\epsilon\to 0} A(\epsilon) = \infty$. This implies

$$u_\infty(x(\alpha), \alpha) = \infty \qquad \forall\, 0 < \alpha < \alpha_0. \tag{4.23}$$

Step 2: Propagation. In order to prove that the blow-up propagates along Γ we have replace $t = 0$ by $t = \alpha < \alpha_0$. We claim that

$$\int_{B_\sigma(x(\alpha))} u_\infty(x, \alpha)dx = \infty \qquad \forall \sigma > 0. \tag{4.24}$$

Actually, it is sufficient to prove the result with $\sigma = \epsilon$ and with u_∞ replaced by v_ϵ. Then

$$\int_{B_\epsilon(x(\alpha)} v_\epsilon(x, \alpha)dx = \int_{B_\epsilon} \tilde{v}_\epsilon(x, \alpha)dx = \epsilon^{N-\frac{2}{p-1}} e^{\frac{\ell(\epsilon)}{p-1}} \int_{B_1} \omega_\epsilon(x, \epsilon^{-2}\alpha)dx \tag{4.25}$$

Using (4.17), (4.21) we have

$$\epsilon^{N-\frac{2}{p-1}} e^{\frac{\ell(\epsilon)}{p-1}} \int_{B_1} \omega_\epsilon(x, \epsilon^{-2}\alpha)dx \geq c_1' e^{A'(\epsilon)} \int_{B_1} \psi_0(x)dx$$

where

$$A'(\epsilon) := (N - \frac{2}{p-1})\ln\epsilon + \frac{\ell(\epsilon)}{p-1} - \left(\lambda_0 + \frac{\beta_\tau^2}{4} + \frac{\delta_\tau}{2} + \sigma_\tau\right)\frac{\alpha}{\epsilon^2} \tag{4.26}$$

Thus $\lim_{\epsilon\to 0} A'(\epsilon) = \infty$ for any $\alpha < \alpha_0$. This implies the claim.

Step 3: End of the proof. For $k > 0$, we denote by $u_{\alpha,k\delta}$ the solution of

$$\left\{\begin{array}{ll} \partial_t u - \Delta u + h(x,t)u^p = 0 & \text{in } \mathbb{R}^N \times (\alpha, \infty) \\ u(.,\alpha) = k\delta_{x(\alpha)} & \text{in } \mathbb{R}^N. \end{array}\right. \tag{4.27}$$

We claim that

$$u_\infty(x,t)) \geq u_{\alpha,k\delta}(x,t) \qquad \forall (x,t) \in \mathbb{R}^N \times [\alpha, \infty). \tag{4.28}$$

We fix $k > 0$, then for any $\sigma > 0$, there exists $m = m(\sigma) > 0$ such that

$$\int_{B_\sigma(x(\alpha))} \min\{m, u_\infty(x, \alpha)\}dx = k.$$

Furthermore $\lim_{\sigma\to 0} m(\sigma) = \infty$. Let $u_{\alpha,k,\sigma}$ be the solution of

$$\left\{\begin{array}{ll} \partial_t u - \Delta u + h(x,t)u^p = 0 & \text{in } \mathbb{R}^N \times (\alpha,\infty) \\ u(.,\alpha) = \min\{m, u_\infty(.,\alpha)\}\chi_{B_\sigma(x(\alpha))} & \text{in } \mathbb{R}^N. \end{array}\right. \tag{4.29}$$

By the maximum principle $u_\infty \geq u_{\alpha,k,\sigma}$ in $\mathbb{R}^N \times (\alpha,\infty)$. But $\min\{m, u_\infty(.,\alpha)\}\chi_{B_\sigma(x(\alpha))}$ converges to $k\delta_{x(\alpha)}$ in the weak sense of measure when $\sigma \to 0$. By stability, since we have assumed $p < 1 + \frac{2}{N}$, $u_{\alpha,k,\sigma} \to u_{\alpha,k\delta}$ locally uniformly in $\mathbb{R}^N \times (\alpha,\infty)$ (see [7] e.g.). Therefore (4.28) follows. Since k is arbitrary, it follows

$$u_\infty(x,t)) \geq u_{\alpha,\infty\delta}(x,t) \qquad \forall (x,t) \in \mathbb{R}^N \times [\alpha,\infty). \tag{4.30}$$

By Step 1, $u_{\alpha,\infty\delta}(x(\alpha+\beta),\alpha+\beta)$ is infinite for $0 < \beta < \alpha_0$. This implies that (4.23) holds for any $0 < \alpha < 2\alpha_0$. Iterating this process we conclude that u_∞ blows-up on whole Γ and (4.24) holds for any $\alpha \in (0,T]$. $\square$

5 Propagation of singularities in the initial plane

In this section, we consider

$$\partial_t u - \Delta u + h(x,t)u^p = 0 \quad \text{in } Q_\infty := \mathbb{R}^N \times (0,\infty) \tag{5.1}$$

where $h \in C(\overline{Q}_\infty)$. We set $x = (x_1,...,x_N) = (x_1,x')$ and we suppose that $h(x,t) > 0$ except when (x,t) belongs to some straight line Γ that we can assume to be the x_1 axis in the plane $t = 0$. We set $d_\infty((x,t),\Gamma) = \max\{\sqrt{t}, |x'|\}$ and write h under the form

$$h(x,t) = e^{-\ell(d_\infty(x,t);\Gamma)} \tag{5.2}$$

where $\ell : (0,\infty) \mapsto (0,\infty)$ is continuous and nonincreasing with limit ∞ at 0.

Proof of Theorem D. Case 1: $1 < p < 1 + \frac{2}{N}$. For $\epsilon > 0$, we consider the "tunnel" with axis Γ defined by

$$T^\epsilon := \{(x,t) : x_1 \in \mathbb{R}, (x',t) \in B'_\epsilon \times (0,\epsilon^2)\}$$

where B'_ϵ is the ball in $\mathbb{R}^{N-1}$ with center 0 and radius ϵ. Since ℓ is decreasing, there holds

$$\partial_t u - \Delta u + e^{-\ell(\epsilon)}u^p \geq 0 \quad \text{in } T_\epsilon.$$

Thus

$$u_{\infty\delta_0} \geq v_{\infty\delta_0} \quad \text{ in } T_\epsilon,$$

where $v_{\infty\delta_0} = \lim_{k\to\infty} v_{k\delta_0}$ and $v_{k\delta}$ is the solution of

$$\left\{\begin{array}{ll} \partial_t v - \Delta v + e^{-\ell(\epsilon)}v^p = 0 & \text{in } T_\epsilon \\ v = k\delta_0 & \text{in } \mathbb{R} \times B'_\epsilon \\ v = 0 & \text{in } \mathbb{R} \times \partial B'_\epsilon. \end{array}\right. \tag{5.3}$$

We put

$$v_{\infty\delta}(x,t) = \epsilon^{-2/(p-1)}e^{\ell(\epsilon)/(q-1)}w(x/\epsilon, t/\epsilon^2)$$

Then $w = w_\epsilon$ satisfies

$$\left\{\begin{array}{ll} \partial_t w - \Delta w + w^p = 0 & \text{in } \mathbb{R} \times B_1' \times (0,1) \\ w = \infty\delta_0 & \text{in } \mathbb{R} \times B_1' \\ w = 0 & \text{in } \mathbb{R} \times \partial B_1'. \end{array}\right. \tag{5.4}$$

We denote by W the solution of

$$\left\{\begin{array}{ll} \partial_\tau W - \Delta W + W = 0 & \text{in } \mathbb{R} \times B_1' \times (0,1) \\ W(\xi_1, \xi', 0) = \psi(\xi_1)\phi(\xi') & \text{in } \mathbb{R} \times B_1' \\ W = 0 & \text{in } \mathbb{R} \times \partial B_1' \times (0,1). \end{array}\right. \tag{5.5}$$

where ϕ is the first eigenfunction of $-\Delta_{\xi'}$ in $W_0^{1,2}(B_1')$ with maximum 1 and corresponding eigenvalue λ and $\psi(\xi_1) = \cos(\xi_1)\chi_{[-\frac{\pi}{2},\frac{\pi}{2}]}(\xi_1)$. Then

$$W(\xi_1, \xi', \tau) = \frac{e^{-(\lambda+1)\tau}\phi(\xi')}{\sqrt{4\pi\tau}} \int_{-\pi/2}^{\pi/2} e^{-|\xi_1 - \zeta|^2/4\tau}\psi(\zeta)d\zeta.$$

Since $0 \leq W \leq 1$, W is a subsolution for the equation satisfied by w and there exists $a > 0$ and $c > 0$ such that

$$w_\epsilon(\xi, \tau + a) \geq cW(\xi, \tau) \qquad \forall (x,t) \in \mathbb{R} \times B_1' \times (0,1).$$

Returning to $v_\infty\delta_0$, we derive

$$v_{\infty\delta_0}(x, t + a\epsilon^2) \geq c\epsilon^{-2/(p-1)} e^{\ell(\epsilon)/(q-1)} W(x/\epsilon, t/\epsilon^2) \qquad \forall (x_1, x', t) \in \mathbb{R} \times B_\epsilon' \times (0, \epsilon^2], \tag{5.6}$$

which implies, with $t = \epsilon^2$ and $x' = 0$,

$$v_\infty(x_1, 0, (a+1)\epsilon^2) \geq c\epsilon^{-2/(p-1)} e^{\ell(\epsilon)/(q-1)} \frac{e^{-\lambda-1}\phi(0)}{\sqrt{4\pi}} \int_{-\pi/2}^{\pi/2} e^{-|x_1/\epsilon - \zeta|^2/4}\psi(\zeta)d\zeta. \tag{5.7}$$

But

$$\int_{-\pi/2}^{\pi/2} e^{-|x_1/\epsilon - \zeta|^2/4}\psi(\zeta)d\zeta \geq e^{-x_1^2/2\epsilon^2} \int_{-\pi/2}^{\pi/2} e^{-|\zeta|^2/2}\psi(\zeta)d\zeta \tag{5.8}$$

If we fix in particular $|x_1| \leq \delta$ where

$$|x_1| < \delta = \sqrt{\frac{2\epsilon^2\ell(\epsilon)}{q-1}},$$

we derive

$$\lim_{\epsilon\to 0} v_\infty(x_1, 0, (a+1)\epsilon^2) = \infty. \tag{5.9}$$

Furthermore this limit is uniform for $x_1 \in [-\delta', \delta']$, where $\delta' < \delta$. Furthermore the interval $[-\delta', \delta']$ does not shrink to $\{0\}$ when $\epsilon \to 0$, since it is assumed that

$$\liminf_{\epsilon\to 0} \epsilon^2\ell(\epsilon) > 0. \tag{5.10}$$

Replacing $[-\delta', \delta']$ by $[\delta', 3\delta']$ and $[-3\delta', -\delta']$ and iterating, we conclude that

$$\lim_{t\to 0} u_{\infty\delta_0}(x_1, 0, t) \geq \lim_{t\to 0} v_\infty(x_1, 0, t) = \infty. \tag{5.11}$$

From this, it is easy to obtain that $u_{\infty\delta_0}(x,t) = u_{\infty\delta_0}(0, x', t)$ is independent of x_1 and coincide with $U(x', t)$ where U is the solution of

$$\left\{\begin{array}{rl} \partial_t U - \Delta U + e^{-\ell(\max\{\sqrt{t}, |x'|\})} U^p = 0 & \text{in } (0,\infty) \times \mathbb{R}^{N-1} \\ U = \infty\delta_0 & \text{in } \mathbb{R}^{N-1} \end{array}\right. \tag{5.12}$$

Case 2: $p \geq 1 + \frac{2}{N}$. We write h under the form

$$h(x,t) = \left(\max\{\sqrt{t}, |x'|\}\right)^\gamma e^{-\tilde{\ell}(\max\{\sqrt{t},|x'|\})} \tag{5.13}$$

where $\tilde{\ell}(s) = \ell(s) - \gamma \ln s$ and $\gamma > N(p-1) - 2$. Then $\gamma > 0$, (1.3) is satisfied and for any $k > 0$ there exists a unique solution $v = v_{k\delta}$ to

$$\left\{\begin{array}{rl} \partial_t v - \Delta v + e^{-\tilde{\ell}(\epsilon)} \left(\max\{\sqrt{t}, |x'|\}\right)^\gamma v^p = 0 & \text{in } T_\epsilon \\ v = k\delta_0 & \text{in } \mathbb{R} \times B'_\epsilon \\ v = 0 & \text{in } \mathbb{R} \times \partial B'_\epsilon. \end{array}\right. \tag{5.14}$$

Furthermore $u_\infty(x,t) \geq v_{\infty\delta}$ in T_ϵ. We set

$$v_{\infty\delta}(x,t) = \epsilon^{-(2+\gamma)/(p-1)} e^{\ell(\epsilon)/(q-1)} w(x/\epsilon, t/\epsilon^2),$$

and $w = w_\epsilon$ satisfies

$$\left\{\begin{array}{rl} \partial_\tau w - \Delta w + \left(\max\{\sqrt{\tau}, |\xi'|\}\right)^\gamma w^p = 0 & \text{in } \mathbb{R} \times B'_1 \times (0,1) \\ w = \infty\delta_0 & \text{in } \mathbb{R} \times B'_1 \\ w = 0 & \text{in } \mathbb{R} \times \partial B'_1 \times (0,1). \end{array}\right. \tag{5.15}$$

We denote by W the solution of

$$\left\{\begin{array}{rl} \partial_\tau W - \Delta W + \left(\max\{\sqrt{\tau}, |\xi'|\}\right)^\gamma W = 0 & \text{in } \mathbb{R} \times B'_1 \times (0,1) \\ W(\xi_1, \xi', 0) = \psi(\xi_1)\phi(\xi') & \text{in } \mathbb{R} \times B'_1 \\ W = 0 & \text{in } \mathbb{R} \times \partial B'_1 \times (0,1). \end{array}\right. \tag{5.16}$$

where ψ and ϕ are as in the first case. This equation admits a separable solution $W(\xi, \tau) = W_1(\xi_1, \tau) W'(\xi', \tau)$ where

$$\left\{\begin{array}{rl} \partial_\tau W' - \Delta_{\xi'} W' + \left(\max\{\sqrt{\tau}, |\xi'|\}\right)^\gamma W' = 0 & \text{in } B'_1 \times (0,1) \\ W'(\xi', 0) = \phi(\xi') & \text{in } B'_1 \\ W' = 0 & \text{in } \partial B'_1 \times (0,1), \end{array}\right. \tag{5.17}$$

and
$$\begin{cases} \partial_\tau W_1 - \partial_{x_1x_1} W_1 + W_1 = 0 & \text{in } \mathbb{R}\times(0,1) \\ W_1(\xi_1,0) = \phi(\xi_1) & \text{in } \mathbb{R} \end{cases} \tag{5.18}$$
Thus
$$W(\xi_1,\xi',\tau) = \frac{W'(\xi',\tau)}{\sqrt{4\pi\tau}}\int_{-\pi/2}^{\pi/2} e^{-|\xi_1-\zeta|^2/4\tau}\psi(\zeta)d\zeta.$$
The exact expression of W' is not simple but since $(\max\{\sqrt{\tau},|\xi'|\})^\gamma \leq 1$ in $B_1'\times(0,1)$, there holds
$$W(\xi_1,\xi',\tau) \geq \frac{e^{-(\lambda+1)\tau}\phi(\xi')}{\sqrt{4\pi\tau}}\int_{-\pi/2}^{\pi/2} e^{-|\xi_1-\zeta|^2/4\tau}\psi(\zeta)d\zeta.$$
From this point, and since (5.10) holds with ℓ replaced by $\tilde{\ell}$, the proof is the same as in Case 1. $\square$

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